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Singular Perturbations of Piecewise Monotonic Maps of the Interval

Abraham Boyarsky[1]

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Let τ: $[0, 1] \to [0, 1]$ be a piecewise monotonic, C^2, and expanding map. In computing an orbit $\{\tau^i(x_0)\}_{i=0}^{\infty}$, we model the roundoff error at each iteration by a singular perturbation; i.e., $x_{n+1} = \tau(x_n) + W_\varepsilon$, where W_ε is a random variable taking on discrete values in an interval $(-\varepsilon, \varepsilon)$. The main result proves that this process admits an absolutely continuous invariant measure which approaches the absolutely continuous measure invariant under the deterministic map τ as the precision of computation $\varepsilon \to 0$.

KEY WORDS: Piecewise monotonic; expanding maps; singular perturbations; absolutely continuous invariant measures; computer orbits.

1. INTRODUCTION

Many mathematical models can be viewed as idealizations of nature and there is often some quantifiable measure of the extent to which the model approximates reality. However, very little is known about how to estimate the fidelity with which numerical simulations reflect the complicated behavior of dynamical systems. (See Chapter 2 of Ref. 14 for a detailed discussion.) What has been observed is that computer simulations seem to yield the "right" results in many cases. More specifically, let τ: $[0, 1] \to [0, 1]$ be a map that admits an invariant measure that is absolutely continuous with respect to Lebesgue measure, as, for example, piecewise monotonic maps that are expanding.[4] It can be verified that for many such τ, the computed orbit of almost every starting point under iterations of τ has a histogram that approximates the histogram obtained from the absolutely continuous invariant measure.

A model for computer orbits is based on the following: we assume that

[1] Department of Mathematics, Loyola Campus, Concordia University, Montreal, Canada H4B 1R6.

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the computation can be done with infinite precision and the roundoff error is added as a noise term at each step. Thus, at the nth step, $x_{n+1} = \tau(x_n) + W$, where W is a random variable. Such a model has been used in Refs. 1, 2, 6, 9, 11, 13, and 16, where the perturbation term is assumed to be "absolutely continuous," i.e., the probability density function of W is supported on intervals. A more general model for the random perturbation term is presented in Ref. 18 for endomorphisms of the interval $[0, 1]$ satisfying the conditions of Misiurewicz.[19] Because such a perturbation term smears computational approximations in a smooth way, they give erroneous predictions in some simple cases. For example, because most computers use binary arithmetic, the calculations for the triangle map $\tau: [0, 1] \to [0, 1]$, defined by $\tau(x) = 2x$, $0 \leqslant x \leqslant \frac{1}{2}$, and $\tau(x) = 2(1-x)$, $\frac{1}{2} \leqslant x \leqslant 1$, are exact for dyadic rationals. Thus, the computed orbit becomes constant at 0 after a few iterations. But the model with a smooth perturbation density predicts a uniform density on $[0, 1]$!

The problem in the above model lies in the assumption that the computation error is uniformly distributed, usually in the range $-\frac{1}{2}$ to $\frac{1}{2}$ times the least significant digit kept. This ignores "the obvious fact that the actual distribution of machine roundoff must be granular since only certain roundoffs can occur" (Ref. 14, p. 33).

In Ref. 15, certain types of singular perturbations are considered. By a singular perturbation in Ref. 15 is meant a continuous function close to the identity. Furthermore, the map τ must be continuous.

In this paper we model the computer error by a finite, discrete random variable, and prove that the absolutely continuous measure of the map τ is stable under such perturbations. This yields *a priori* assurance that computer orbits of certain maps will exhibit the correct invariant measure.

Most of the ideas in this paper are not new; the main observation is that they continue to work when the perturbations are "singular." For example, Theorem 1 is well known and the ideas for the proof of the main theorem are mostly borrowed from Ref. 16.

For interesting discussions on computer orbits, the reader is referred to Refs. 6–8 and 10.

2. SINGULAR PERTURBATION COMPUTER MODEL

Let $I = [0, 1]$ and let $(\mathscr{L}_1, \|\cdot\|_1)$ denote the space of all integrable functions with respect to Lebesgue measure on I. Let $\tau: I \to I$ be a nonsingular transformation. The Frobenius–Perron operator[5] $P_\tau: \mathscr{L}_1 \to \mathscr{L}_1$ is defined by

$$P_\tau f(x) = \frac{d}{dx} \int_{\tau^{-1}[0,x]} f(y)\, dy$$

If X is a random variable with probability density function f, then $P_\tau f$ can be interpreted as the probability density function of the random variable $\tau(X)$.

Let $x \in I$. In computing $\tau(x)$, we shall regard the number computed as the true answer plus a random number (Ref. 14, Section 2.5), i.e.,

$$x_{n+1} = \tau(x_n) + W_\varepsilon \tag{1}$$

Since the error between the true value $\tau(x_n)$ and the computed value x_{n+1} must differ by a number from a discrete set of numbers, W_ε is a discrete random variable. Let ε denote the magnitude of the largest computer error that can occur. It is a measure of the precision of computation. Thus, W_ε is a discrete random variable with the values it can assume restricted to the set $[-\varepsilon, \varepsilon]$.

Let $\delta_a(x)$ denote the point measure at $a \in I$. Then we can write the probability function of W_ε as

$$h_\varepsilon(x) = \sum_{i=1}^{N_\varepsilon} \lambda_i^\varepsilon \delta_{a_i^\varepsilon}(x), \qquad \sum_{i=1}^{N_\varepsilon} \lambda_i^\varepsilon = 1, \qquad \lambda_i^\varepsilon \geqslant 0$$

where $a_i^\varepsilon \in [-\varepsilon, \varepsilon]$, λ_i^ε denotes the probability that the error will be a_i^ε, and N_ε is the total number of computer errors that can occur.

It is reasonable to assume that the perturbation random variable is independent of x_n. Hence, if f is the probability density function of x_n in (1), then the probability density function of x_{n+1} is obtained by convoluting the probability density function of $\tau(x_n)$, $P_\tau f$, and $h_\varepsilon(x)$:

$$\mathscr{P}_\varepsilon f(x) = \int_0^1 P_\tau f(x-z) \sum_{i=1}^{N_\varepsilon} \lambda_i^\varepsilon \delta_{a_i^\varepsilon}(z)\, dz$$

i.e.,

$$\mathscr{P}_\varepsilon f(x) = \sum_{i=1}^{N_\varepsilon} \lambda_i^\varepsilon P_\tau f(x - a_i^\varepsilon) \tag{2}$$

Let $a \in R$, and let $\sigma_a \colon R \to R$ be defined by $\sigma_a(x) = x + a$. Then the Frobenius–Perron operator of σ_a is

$$P_{\sigma_a} f(x) = \frac{d}{dx} \int_{-\infty}^{x-a} f(y)\, dy = f(x-a)$$

where $P_{\sigma_a} \colon \mathscr{L}_1(R) \to \mathscr{L}_1(R)$.

Thus, (2) can be written as

$$\mathscr{P}_\varepsilon f(x) = \sum_{i=1}^{N_\varepsilon} \lambda_i^\varepsilon P_{\sigma_{a_i^\varepsilon}} P_\tau f(x)$$

It is easy to show that $P_\sigma P_\tau = P_{\sigma \circ \tau}$. Thus,

$$\mathscr{P}_\varepsilon f(x) = \sum_{i=1}^{N_\varepsilon} \lambda_i^\varepsilon P_{\tau_i^\varepsilon} f(x) \tag{3}$$

where $\tau_i^\varepsilon = \sigma_{a_i^\varepsilon} \circ \tau$. Since $\sigma_{a_i^\varepsilon}$ shifts τ by a_i^ε, τ_i^ε will not be defined on a small interval either to the right of 0 or to the left of 1, as shown in Fig. 1. This reflects the possibility that if x_n is close to 0 or 1, the perturbation will take the computer orbit out of I. Since this does not happen in practice, we shall extend τ_i^ε in such a way as to guarantee that the orbit stays in I. In the sequel we shall be considering only piecewise monotonic maps that are expanding. To keep the extension in this class, we shall extend τ_i^ε to $\bar{\tau}_i^\varepsilon$ on all of I by redefining τ_i^ε on a small interval near 0 or 1, depending on whether a_i^ε is negative or positive.

A map $\tau: I \to I$ is called *piecewise monotonic* and C^2 if there is a partition of I, $0 = c_0 < c_1 < \cdots < c_k = 1$, so that for each $i = 0, 1,..., k-1$, $\tau|_{(c_i, c_{i+1})}$ is monotone and extends to a C^2 map on $[c_i, c_{i+1}]$. Under the expansiveness condition $\inf |\tau'| > 1$, it is shown in Ref. 4 that τ admits an absolutely continuous invariant measure μ. Let $l_i = c_{i+1} - c_i$ denote the length of the ith interval.

Let $\lambda_0 \equiv \inf |\tau'(x)| > 1$. Without loss of generality consider the case $a_i^\varepsilon > 0$. Assume $\tau(1) = 1$. Then $\tau_i^\varepsilon(1 - a_i^\varepsilon) = 1$. We would like to define τ_i^ε on all of I in such a way as to keep the same number of monotonic pieces as τ

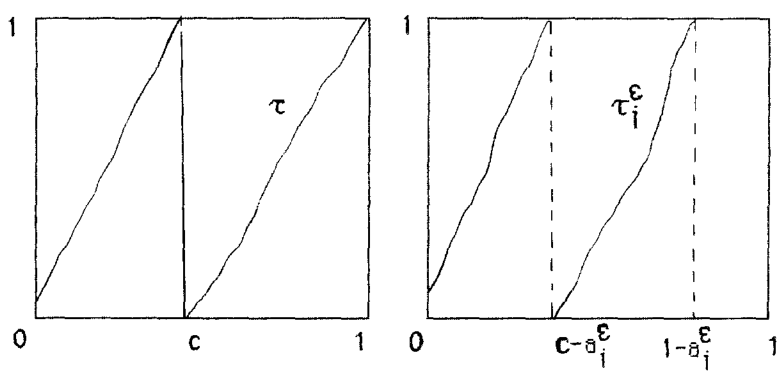

Figure 1

and such that $\inf_I |(\tau^\varepsilon)'| > 1$. Fix $\varepsilon_0 > 0$ and let $\varepsilon < \varepsilon_0$. Define an integer $M = M(\varepsilon_0) > 0$ such that

$$\frac{1 - \tau(1 - M\varepsilon)}{M\varepsilon} \geqslant \lambda > 1 \tag{4}$$

where $\lambda \leqslant \lambda_0$ is fixed and independent of ε. Clearly $M(\varepsilon_0) \to 0$ as $\varepsilon_0 \to 0$. Now define $\bar{\tau}_i^\varepsilon$ as follows:

$$\bar{\tau}_i^\varepsilon(x) = \begin{cases} \tau_i^\varepsilon(x), & 0 \leqslant x \leqslant 1 - M\varepsilon \\ \text{smooth, expanding continuation of last piece of} \\ \tau_i^\varepsilon(x) \text{ to } [1 - M\varepsilon, 1] \end{cases}$$

That smooth, expanding continuation is possible follows from (4). Figure 2 shows the extension for the example in Fig. 1.

If $0 < \tau(1) < 1$, then for sufficiently small ε_0, the last segment of τ can be continued as above, directly from $(a_i^\varepsilon, \tau_i^\varepsilon(1 - a_\varepsilon))$. The foregoing extension applies if $M\varepsilon_0 < l_k$, the length of the last interval on which the partition is monotonic. Otherwise we choose ε_0 small enough so that $c_{k-1} < 1 - M\varepsilon_0 < c_k$, where M is defined by (4).

If $\tau(1) = 0$, or if $a_i^\varepsilon < 0$, an extension analogous to the one above is performed. Thus, for each $a_i^\varepsilon \in (-\varepsilon, \varepsilon)$, $\varepsilon < \varepsilon_0$, ε_0 sufficiently small: (i) $\bar{\tau}_i^\varepsilon$ has the same number of monotonic pieces as τ, and (ii) $\inf |(\tau_i^\varepsilon)'| \geqslant \lambda > 1$ $\forall \varepsilon < \varepsilon_0$, where λ is a fixed number $\leqslant \lambda_0$.

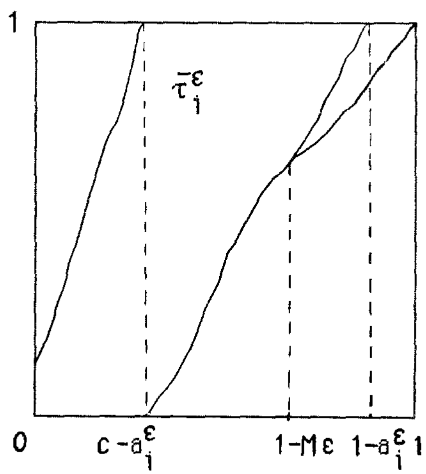

Figure 2

Note that the extension near the edges is invoked only when the computer orbit is at the computer numbers that are close to 0 or 1.

Let

$$\bar{\mathscr{P}}_\varepsilon f(x) = \sum_{i=1}^{N_\varepsilon} \lambda_i^\varepsilon P_{\bar{\tau}_i^\varepsilon} f(x)$$

Then $\bar{\mathscr{P}}_\varepsilon f$ denotes the probability density function of the $(n+1)$th computer iterate, where f is the probability density function of the nth computer iterate.

In the terminology of Ref. 3, $\bar{\mathscr{P}}_\varepsilon$ is the Frobenius–Perron operator of a random map, where one of $\bar{\tau}_1^\varepsilon, \bar{\tau}_2^\varepsilon, ..., \bar{\tau}_{N_\varepsilon}^\varepsilon$ is applied at any time with respective probabilities $\lambda_1^\varepsilon, \lambda_2^\varepsilon, ..., \lambda_{N_\varepsilon}^\varepsilon$.

Theorem 1. Let $\tau: I \to I$ be piecewise monotonic and C^2 and expanding. Then for any $\varepsilon > 0$, $\bar{\mathscr{P}}_\varepsilon$ admits an absolutely continuous invariant measure μ_ε.

Proof. By definition of $\bar{\tau}_i^\varepsilon$, $\bar{\tau}_i^\varepsilon$ is expanding. Let $\alpha_i^\varepsilon \equiv \inf |\bar{\tau}_i^\varepsilon| > 1$. Hence

$$\sum_{i=1}^{N_\varepsilon} \frac{\lambda_i^\varepsilon}{|\tau_i^{\varepsilon'}|} < \sum_{i=1}^{N_\varepsilon} \lambda_i^\varepsilon < 1$$

Thus, Theorem 1 of Ref. 3 applies, establishing the existence of an absolutely continuous invariant measure μ_ε. ∎

3. CONTINUITY OF INVARIANT MEASURES

The main result of this section is to prove that the absolutely continuous measure invariant under the computer model whose operator is $\bar{\mathscr{P}}_\varepsilon$ approaches the absolutely continuous measure invariant under τ.

Lemma 1. Consider the random map $\bar{\tau}^\varepsilon$: $\bar{\tau}_1^\varepsilon, \bar{\tau}_2^\varepsilon, ..., \bar{\tau}_{N_\varepsilon}^\varepsilon$ with respective probabilities $\lambda_1^\varepsilon, \lambda_2^\varepsilon, ..., \lambda_{N_\varepsilon}^\varepsilon$, $\lambda_i^\varepsilon \geqslant 0$, $\sum_{i=1}^{N_\varepsilon} \lambda_i^\varepsilon = 1$, $\varepsilon \leqslant \varepsilon_0$, where τ is a piecewise monotonic, C^2, expanding map. Then there exist constants α and β such that for $f \in \mathscr{L}_1(I)$

$$\bigvee_0^1 P_{\bar{\tau}_i^\varepsilon} f \leqslant \alpha \, \|f\|_1 + \beta \bigvee_0^1 f \tag{5}$$

for all $i = 1, ..., N_\varepsilon$, and $\forall \varepsilon \leqslant \varepsilon_0$, where $\bigvee_0^1$ denotes the variation over I.

Proof. By construction, $\bar{\tau}_i^\varepsilon$ has the same number of monotonic pieces as τ, namely k, and $\inf |(\bar{\tau}_i^\varepsilon)'| \geqslant \lambda > 1$, independent of i and ε. Let $0 = c_0^\varepsilon <$

$c_1^\varepsilon < \cdots < c_k^\varepsilon = 1$ denote the partition points of $\bar{\tau}_i^\varepsilon$, i.e., $\bar{\tau}_i^\varepsilon|_{(c_j^\varepsilon, c_{j+1}^\varepsilon)}$ is monotonic, and let $l_j^\varepsilon = c_{j+1}^\varepsilon - c_j^\varepsilon$. By definition of $\bar{\tau}_i^\varepsilon$, $\min_j l_j^\varepsilon \equiv l > 0$ $\forall \varepsilon \leqslant \varepsilon_0$. In the proof of Theorem 1 of Ref. 4, it is shown that $\exists \alpha_i^\varepsilon, \beta_i^\varepsilon \ni$

$$\bigvee_0^1 P_{\bar{\tau}_i^\varepsilon} f \leqslant \alpha_i^\varepsilon \|f\|_1 + \beta_i^\varepsilon \bigvee_0^1 f$$

Now β_i^ε depends only on $\inf |(\bar{\tau}_i^\varepsilon)'|$. Since $\inf |(\bar{\tau}_i^\varepsilon)'| \geqslant \lambda$ for all i and for all $\varepsilon \leqslant \varepsilon_0$, β_i^ε is independent of i and ε. The constant α_i^ε is given by

$$\alpha_i^\varepsilon = K_i^\varepsilon + \frac{2}{\min_j l_j^\varepsilon}$$

where $K_i^\varepsilon = \max |\sigma_i^{\varepsilon\prime}| / \min \sigma_i^\varepsilon$, and σ_i is defined by $\phi_i^\varepsilon = (\tau_i^\varepsilon)^M$, where M is such that $|(\phi_i^\varepsilon)'| > 2$; $\sigma_i^\varepsilon(x) = |((\phi_i^\varepsilon)^{-1})'(x)|$. It is easy to show that $\exists K$ such that $K_i^\varepsilon \leqslant K$ $\forall i$, $\forall \varepsilon \leqslant \varepsilon_0$. By the definition of l, we obtain

$$\alpha_i^\varepsilon \leqslant K + 2/l \equiv \alpha \quad \blacksquare$$

Following Ref. 16, we introduce a metric on the class of piecewise monotonic, expanding transformations:

$$\begin{aligned} d(\tau_1, \tau_2) \equiv \inf\{\delta > 0 \colon & \exists A \subseteq I, \exists \sigma \colon I \to I \ni m(A) > 1 - \delta, \\ & \sigma \text{ is a diffeomorphism, } \tau_1|_A = \tau_2 \circ \sigma|_A, \text{ and} \\ & \forall x \in A \colon |\sigma(x) - x| < \delta, \ |1/\sigma'(x) - 1| < \delta\} \end{aligned}$$

where m is Lebesgue measure on I.

Lemma 2. Let $\bar{\tau}_\varepsilon$ be the random map above derived from the piecewise monotonic, C^2, expanding map τ. Then

$$d(\bar{\tau}_i^\varepsilon, \tau) \leqslant 2M\varepsilon$$

Proof. By the definition of $\bar{\tau}_i^\varepsilon$, $\bar{\tau}_i^\varepsilon(x) = \tau(x - a_i^\varepsilon)$ for $x \in [0, 1 - M\varepsilon]$ if $a_i^\varepsilon > 0$, or for $x \in [M\varepsilon, 1]$ if $a_i^\varepsilon < 0$. Without loss of generality, we consider the first case. Let σ_i^ε be the shift map: $R \to R$ defined, as earlier, by $\sigma_i^\varepsilon(x) = x + a_i^\varepsilon$. Since $|a_i^\varepsilon| < \varepsilon$, $|\sigma_i^\varepsilon(x) - x| < \varepsilon$. Obviously, $|(\sigma_i^\varepsilon)'(x)| = 1$. To make $\sigma_i^\varepsilon|_I$ a diffeomorphism, we redefine it to the left of 1 and to the right of 0 on a set of measure $M\varepsilon$ so that

$$\bar{\sigma}_i^\varepsilon(x) = \begin{cases} 0, & \text{if } x = 0 \\ \sigma_i^\varepsilon(x), & \text{if } M\varepsilon < x < 1 - M\varepsilon \\ 1, & \text{if } x = 1 \end{cases}$$

and $\bar{\sigma}_i^\varepsilon(x)$ is a diffeomorphism on I.

Let $A=(M\varepsilon, 1-M\varepsilon)$. Then $m(A)=1-2M\varepsilon$, and $\bar{\tau}_i^\varepsilon|_A=\tau\circ\bar{\sigma}_i^\varepsilon|_A$. Hence, $d(\bar{\tau}_i^\varepsilon, \tau)\leqslant 2M\varepsilon$. ▌

Definition.[16] Let $BV=\{f\in\mathscr{L}_1: \bigvee_0^1 f<\infty\}$ and for $f\in BV$, define $\|f\|_V\equiv\bigvee_0^1 f+\|f\|_1$. For an operator $P: BV\to\mathscr{L}_1$, define the norm

$$|||P|||=\sup\{\|Pf\|_1: f\in BV, \|f\|_V\}\leqslant 1$$

Lemma 3:

$$|||P_\tau-\bar{\mathscr{P}}_\varepsilon|||\leqslant 12\max_{i=1,\dots,N_\varepsilon} d(\tau, \bar{\tau}_i^\varepsilon)$$

Proof. By Lemma 13 of Ref. 16, we have

$$|||P_\tau-P_{\bar{\tau}_i^\varepsilon}|||\leqslant 12d(\tau, \bar{\tau}_i^\varepsilon)$$

Now

$$P_\tau-\bar{\mathscr{P}}_\varepsilon=\sum_{i=1}^{N_\varepsilon}\lambda_i^\varepsilon(P_\tau-P_{\bar{\tau}_i^\varepsilon})$$

Hence

$$\begin{aligned}|||P_\tau-\bar{\mathscr{P}}_\varepsilon|||&\leqslant 12\sum_{i=1}^{N_\varepsilon}\lambda_i^\varepsilon d(\tau, \bar{\tau}_i^\varepsilon)\\&=12\max_{i=1,\dots,N_\varepsilon} d(\tau, \bar{\tau}_i^\varepsilon)\end{aligned}$$ ▌

Let

$$\Phi(1, P)(f)=\lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}P^jf$$

Theorem 2. Let $\tau: I\to I$ be a piecewise monotonic, C^2, and expanding map which admits a unique, absolutely continuous, invariant measure μ, with density f. Then for ε sufficiently small the random map $\bar{\tau}_\varepsilon$ admits a unique, absolutely continuous, invariant measure μ with density f_ε and

$$\|f_\varepsilon-f\|_1\to 0$$

as $\varepsilon\to 0$. Also, $(\bar{\tau}_\varepsilon, f_\varepsilon)$ is mixing for sufficiently small ε.

Proof. By Lemma 1, $\bar{\mathscr{P}}_\varepsilon$ is ζ-bounded, in the terminology of Ref. 16. By Lemmas 2 and 3, $\lim_{\varepsilon\to 0}|||P_\tau-\bar{\mathscr{P}}_\varepsilon|||=0$. Therefore, Theorem 8 of Ref. 16

implies that $(\bar{\tau}_\varepsilon, f_\varepsilon)$ is ergodic for sufficiently small ε and that $\| f_\varepsilon - f \|_1 \to 0$ as $\varepsilon \to 0$.

In Ref. 12 it is shown that (τ, f) is mixing. Hence, it follows from Theorem 8c of Ref. 16 that $(\bar{\tau}_\varepsilon, f_\varepsilon)$ is mixing for sufficiently small ε. ∎

Remark 1. In Refs. 5 and 17 sufficient conditions are given for (τ, μ) to be exact. Then μ is unique and mixing.

Remark 2. With more care Theorem 2 can be proved for the case where τ has l independent invariant densities $f_1, f_2,..., f_l$ and $S_1, S_2,..., S_l$ are their respective supports. Considering the map $\tau|_{S_1}$, we would proceed as above.

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